Journal of Acoustical Society of America

Review for JASA Editor

Title: "The elastic moduli that best approximate the acoustical properties of an anisotropic material"
Manuscript type: Regular Research Article
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Manuscript Number: MS #05-02603

The main interest of the paper is to make the link between the Euclidean projection approach of elastic tensors and the minimization of distance for slowness surfaces and to demonstrate their equivalence while seeking high symmetry approximations. It shows also an alternative form of the basis vector from Walpole for the projection method of the elastic moduli. The author's approach allows to show some interesting properties of the fourth order elastic tensor and of its relationship with the acoustical tensor. I am slightly disappointed not to see any figure or scheme. Discussion with some applications or implications as well as a conclusion and aknowledgements are lacking.

General comments

As a general comment, the paper is dealing with a lot of different mathematical notations. I appreciate the precision of defining the notations. Some are really interesting and useful for simplifying the statements that follow as the operation * on the fourth order tensor. These mathematics can not be avoided but need to be clarified and very well organized. Sometimes a physical or mathematical explanation in the text would be nice so that the reader can follow the different steps more easily (see detailed comments). Especially, I recommend to insert some practical examples with curves or some schemes explaining the concepts so that the problem can have a visual explanation.

For the title and running title, using "Best acoustical fit" is not precise enough. The title has to show the fact a higher symmetry approximation is seeked. The title could be " Elastic moduli approximation of higher symmetry for the acoustical properties of an anisotropic material".

As the subject is complex, the different concepts and mathematical properties are hard to understand at first. Maybe the author should be more careful about the chronology of the different concepts he introduces. When a new concept or mathematical property is introduced, the author should give more detailed explanations at once and not add other properties of the same object afterwhile. I recommend a more carefully organized material especially while introducing each different mathematical definition and properties and more sentences explaining the steps and the meanings. In my opinion, the developments are hard to follow because also of the complexity of the topics itself. A reorganization is needed in clarifying the main interests : the equivalence of the minimization, the projection with the basis vectors of each symmetry from Walpole and the operator *.

In the abstract, neither the definition of " best ", " optimal " nor the norm used are given. The author speaks about implications but does not show any of these nor any applications.

Manuscript detailed comments

1 Introduction

References as [2] and [3] are maybe too specific for a general introduction in the second sentence in page 2. The sentence " Applications of interest are ... materials for which a perfect crystal is not the best analogy " should be changed as the problem is more related to the level of symmetry of the material. The third sentence " ... that do not fit with the presumed symmetry of the material " is also somehow meaningless as the fit depend on the symmetry assumption. The problem is more as Helbig says in [2] the fact one prefers to deal with less parameters.
The citation "Chevrot and Browaeys" according to the references should be "Browaeys and Chevrot".

In " Other relevant works are [6,2,7,8] " needs some explanations about the fact they are relevant. Why is there some quotes for the word distance ? Pointing out the metrics problem (Euclidean, Riemannian, log-Euclidean) is very interesting but this drawback is not solved in this paper, why speaking about it ? The introduction should be clearer about the purpose and the applications of the paper material. It should explain the fact the minimization of the distance working on the elastic tensor or on the acoustical tensor is the same can implies some interesting properties if any.

2 Preliminaries

2.1 The elasticity tensor and related notation

The summation convention is used on scalars and introduced at text note 1 in page 2. When the author write in page 3

$$\mathbf{A} = A_{ij}\mathbf{e}_i\mathbf{e}_j \tag{1}$$

this has to be understood as a tensorial product, I suppose. A short sentence clarifying this would be useful so that the reader does not start being confused.
At the top of page 4, "The fundamental isotropic second order tensors, ... and the tensor $\mathbb{J}$ ... " is not a correct sentence (no verb for the tensor $\mathbb{J}$). An appendix or a more developed and clear paragraph on the properties of $\mathbb{J}$ and $\mathbb{I}$ might be welcomed as they are useful in the following of the article, for example for checking the definition of the inner product in (25), the symmetry of this inner product, the equation of an isotropic elastic tensor $\mathbb{C}_{sym} = \alpha_1\mathbb{I} + \alpha_2\mathbb{J}$ and in equation (24).
Before equation (7), the positive energy implies already (2), i.e. $c_{IJ} = c_{JI}$ or $C_{ijkl} = C_{klij}$. At the bottom of page 4, I guess it is " 6-dimensionnal space of asymmetric tensor".
In page 5, while reading " The partition of $\mathbb{C}$ as a sum of totally ... is related to Backus' harmonic decomposition ", is it not exactly the first step of Backus' decomposition ? Equation (12) is elegant.

2.2 The acoustical tensor and the tensor $\mathbb{C}^*$

The acoustical tensor $\mathbf{Q}$ is also known as the Christoffel's matrix and (13) is also called the Christoffel's equation. In page 5, the eigenvectors are the polarization vectors $\mathbf{u}_0$ and none at all the phase propagation directions $\mathbf{n}$. Equation (15), according to author's notation should be written with C^*_{ijkl} and not with $\mathbb{C}^*_{ijkl}$. Equations (15) and (16) are really interesting and fundamentals. In page 6, " with taking the symmetric and symmetric " should be, I suppose, " with taking the symmetric and asymmetric ". The author should tell also the * operation is a linear operator even if it is obvious, it helps the reader to be reminded of that. Before (19) there are two "iff" instead of "if". The author can tell that the decomposition of $\mathbb{C}$ in a symetric and asymetric part is unique so that the knowledge of $\mathbb{C}^{(s)}$ and $\mathbb{C}^{(a)}$ is equivalent to the one of $\mathbb{C}$ or the one of $\mathbb{C}^*$.

3 Fedorov's problem for particular symmetries

3.1 Definition of the problem

Equation (21) should be written more precisely with the two spherical angles instead of the compact form with $\mathrm{d}\Omega$. In equation (22), how exactly is a determined ?

3.2 The elastic projection

Equation (32) is the method already used by Arts [6]. Note in (33) that if the vectors $\mathbb{V}_i$ constitute an orthonormal basis, $\mathbf{\Lambda}$ is the identity. The author should tell in one or two sentences the comparison in this section of this projector with the previous methods in [2] [4] and [6]. The author should be more precise about the structure of the vector space of $\mathbb{V}_i$. Equation (37) is a very interesting property. At the end of the section "a more more" means "a more".

3.3 Solution of the generalized Fedorov problem

Equation (38) seems correct but implies the stability of the space defined for the given the symmetry while applying the operator *. Explicitly say that if

$$\mathbb{C}_{sym} = \sum_{i=1}^{N} \beta_i \mathbb{V}_i \tag{2}$$

then

$$\mathbb{C}^*_{sym} = \sum_{i=1}^{N} \beta_i \mathbb{V}^*_i = \sum_{i=1}^{N} \alpha_i \mathbb{V}_i \tag{3}$$

and give the explanation and proof.
For the second equation (41), some more explanations about the matrix $\mathbf{R}$ would be helpful as it corresponds to the transformation from the inner product $\langle \mathbb{A}, \mathbb{B} \rangle$ to $\langle \mathbb{A}, \mathbb{B} \rangle_a$ (see also the comments about the properties of $\mathbb{I}$ in section 2.1). This should be introduced while first defining the inner product $\langle \mathbb{A}, \mathbb{B} \rangle_a$ in section 3.1. The fact $\mathbf{R}$ is invertible should be more explained.

$$\langle \mathbb{V}_i, \mathbb{C}^* \rangle_a = \langle \mathbb{V}_i, \mathbb{I}^{(s)} \mathbb{C}^* \rangle = \sum_{j=1}^{N} R_{ij} \langle \mathbb{V}_i, \mathbb{C}^* \rangle \tag{4}$$

$$\sum_{k=1}^{N} R_{ik} \Lambda_{kj} = \langle \mathbb{V}_i, \mathbb{V}_j \rangle_a \tag{5}$$

$$\sum_{j=1}^{N} R_{ij} \langle \mathbb{V}_j, \mathbb{C}^* \rangle = \langle \mathbb{V}_i, \mathbb{C}^* \rangle_a \tag{6}$$

Appendix A

The first equation of (A.2) means the ($\mathbb{V}_1, \ldots, \mathbb{V}_N$) is stable while performing the ^{ast} operation but this has to be demonstrated and stressed even if it is quite straightforward. The matrix $\mathbf{P}$ is invertible because it represents the * operation which is a bijection (in equation (18) page 6). But all these properties should be explained in a paragraph about the * operation when it is introduced. Equation (A.4) is nice and introduced in a nice way the examples of $\mathbf{P}$ in Appendix B. Equation (A.7) means that $\mathbf{P}$ is a very specific matrix that shoulkd be stressed in the text.

Appendix B

The notation diag(1,5) has to be explained. What is the point of giving matrix $\mathbf{P}$ here ? The explicit solution for the projection should be a compact form of the expressions given in [4] and [6]. Could the author be more precise about this ? I recommend maybe to make an appendix B about the general properties of $\mathbf{P}$ and its explicit expressions and another one about the projection in the Walpole basis.